HOMOLOGICAL PERTURBATION THEORY FOR ALGEBRAS OVER OPERADS

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Abstract. We extend homological perturbation theory to encompass algebraic structures governed by operads and cooperads. Specifically, for an operad $\mathcal{O}$, we define the notion of an '$\mathcal{O}$-algebra contraction' and we prove that the formulas of the Basic Perturbation Lemma preserve $\mathcal{O}$-algebra contractions. Over a ground ring containing the rational numbers, we give explicit formulas for constructing an $\mathcal{O}$-algebra contraction from any given contraction, generalizing the so called 'Tensor Trick'.

As an illustration of our results we use them to give short proofs of the transfer and minimality theorems for $\mathcal{O}_\infty$-algebras, where $\mathcal{O}$ is any Koszul operad. This subsumes, but is not restricted to, the cases of A_∞, C_∞ and L_∞-algebras.

Introduction

The goal of this paper is to extend homological perturbation theory to encompass algebraic structures governed by operads and cooperads and to apply the theory to give short proofs of the transfer and minimality theorems for $\mathcal{O}_\infty$-algebras, where $\mathcal{O}$ is any Koszul operad.

Since their advent, *operads* have been a tool to handle homotopy invariant structures, see [4],[32],[1],[31]. For operads $\mathcal{O}$ in chain complexes, one has the notion of a minimal cofibrant resolution $\mathcal{O}_\infty$, algebras over which are called 'strongly homotopy' $\mathcal{O}$-algebras. $\mathcal{O}_\infty$-algebra structures are homotopy invariant in the sense that they can be transferred along homotopy equivalences, see [30]. In general, the minimal cofibrant resolution $\mathcal{O}_\infty$ is difficult to construct, but *Koszul operads* [12],[11],[10],[28], are a class of operads where $\mathcal{O}_\infty$ can be described explicitly. The operads $\mathcal{A}s$, $\mathcal{C}om$ and $\mathcal{L}ie$, governing respectively associative, commutative and Lie algebras, are examples of Koszul operads, and the corresponding strongly homotopy algebras are respectively A_∞ [35], C_∞ [11, §5.3] and L_∞-algebras [26].

For computations, explicit formulas for transferring $\mathcal{O}_\infty$-algebra structures are desirable. Kontsevich and Soibelman [25], based on [33] and [17], wrote down explicit tree formulas for transferring A_∞-algebra structures, and remarked that their formulas should work also for algebras over more general operads. That they work for C_∞-algebras was verified in [7]. Huebschmann [20] has recently given an account of how, in the A_∞ case, the tree formulas can be *deduced* using *homological perturbation theory*. Homological perturbation theory is a set of tools in algebraic topology and homological algebra for handling 'perturbations' of chain complexes, and its origins can be traced to [8],[34],[5],[6],[13]. See [21] for a recent survey with an extensive list of references. As has been acknowledged for quite some time, [16,

This is a reworked version of a part of the author's Ph.D. Thesis [3].

Remark, end of §2.2],[23],[20, Remark 12.2], homological perturbation theory has suffered from the defect of not handling well algebraic structures where symmetries play a role, such as commutative or Lie algebras. For this reason, it has been far from obvious how to use homological perturbation theory to transfer general $\mathcal{O}_\infty$-algebra structures along the lines of [20]. The special case of L_∞-algebras, dealt with recently in [18],[19], required a substantial amount of extra work.

The goal of the present paper is to remove the abovementioned defect by finding a generalization of the classical notion of an *algebra contraction* [22, §2],[16, §2.2] (recalled in Definition 1.5) that works for algebras over any operad. See below for our proposed definition. Our main technical results, which provide generalizations of the 'Algebra Perturbation Lemma' [22, (2.1*)] and the 'Tensor trick' [14, 3.2],[16, §3],[22, (2.2.0*)] (recapitulated in Section 1) are Theorem A and Theorem B below, and their coalgebraic counterparts.

Recall that a *contraction* is a diagram of chain complexes over a commutative ring $\Bbbk$

$$\mathscr{D}\colon \begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}$$

where f and g are morphisms with $fg = 1_B$, where h is a chain homotopy from gf to 1_A, and where $fh = 0$, $hg = 0$, $hh = 0$. If furthermore A and B are equipped with bounded below exhaustive filtrations which are preserved by f, g and h then $\mathscr{D}$ is called a *filtered* contraction. Here is our proposed definition of an $\mathcal{O}$-algebra contraction.

Definition ($\mathcal{O}$-algebra contraction, Definition 6.7)**.** Let $\mathcal{O}$ be an operad. A *contraction of $\mathcal{O}$-algebras* is a contraction $\mathscr{D}$ in which A and B are $\mathcal{O}$-algebras, f and g are morphisms of $\mathcal{O}$-algebras, and there exists a sequence of homotopies $\{h_n\colon A^{\otimes n} \to A^{\otimes n}\}_{n\geq 1}$ with $h_1 = h$ such that

(1) For each $n \geq 1$, the diagram

$$\mathscr{D}_n\colon \begin{tikzcd} A^{\otimes n} \arrow[loop left, "h_n"] \arrow[r, shift left, "f^{\otimes n}"] & B^{\otimes n} \arrow[l, shift left, "g^{\otimes n}"] \end{tikzcd}$$

is a contraction.

(2) For all $\mu \in \mathcal{O}(n)$, the induced operation $\mu_A\colon A^{\otimes n} \to A$ satisfies

$$\mu_A h_n = (-1)^{|\mu|} h \mu_A.$$

(3) For all $p, q \geq 1$, there are equalities of maps from $A^{\otimes p+q}$ to itself

$$(h_p \otimes 1 - 1 \otimes h_q)h_{p+q} = -h_{p+q}(h_p \otimes 1 - 1 \otimes h_q) = h_p \otimes h_q.$$

We say that $\mathscr{D}$ is a *filtered contraction of $\mathcal{O}$-algebras* if, in addition, A and B are equipped with bounded below exhaustive filtrations such that $\mathscr{D}_n$ is a filtered contraction for all n, where $A^{\otimes n}$ and $B^{\otimes n}$ are given the induced filtrations.

Before relating this definition to the classical definition of an algebra contraction, let us state our main results.

A *perturbation* of a chain complex A is a map $t\colon A \to A$ of degree -1 such that $(d_A + t)^2 = 0$ where d_A is the differential in A. Let A^t denote the chain complex A with new differential $d_A + t$.

Theorem A ($\mathcal{O}$-algebra Perturbation Lemma, Theorem 6.8)**.** *Suppose that $\mathscr{D}$ is a filtered contraction of $\mathcal{O}$-algebras. If t is a perturbation of A which lowers the filtration and which is a derivation of $\mathcal{O}$-algebras, then the formulas of the Basic Perturbation Lemma (*[6],[13]*):*

$$\begin{tikzcd} A^t \arrow[loop left, "h'"] \arrow[r, shift left, "f'"] & B^{t'} \arrow[l, shift left, "g'"] \end{tikzcd}, \qquad \begin{array}{ll} f' = f + f\Sigma h, & g' = g + h\Sigma g, \\ h' = h + h\Sigma h, & t' = f\Sigma g, \end{array}$$

where $\Sigma = \sum_{n\geq 0} t(ht)^n$, define a new filtered contraction of $\mathcal{O}$-algebras. In particular, f' and g' are morphisms of $\mathcal{O}$-algebras and the perturbed differential $d_B + t'$ on B is a derivation of $\mathcal{O}$-algebras.

Theorem B ($\mathcal{O}$-algebra Tensor Trick, Theorem 6.10)**.** *Suppose that the ground ring $\Bbbk$ contains $\mathbb{Q}$. Given any contraction $\mathscr{D}$ there is a filtered contraction of $\mathcal{O}$-algebras*

$$\mathcal{O}[\mathscr{D}]\colon\ \begin{tikzcd} \mathcal{O}[A] \arrow[loop left, "H"] \arrow[r, shift left, "F"] & \mathcal{O}[B] \arrow[l, shift left, "G"] \end{tikzcd}$$

where $\mathcal{O}[A]$ denotes the free $\mathcal{O}$-algebra on A, where F and G are the morphisms of $\mathcal{O}$-algebras induced by f and g, and where H is induced by the 'symmetrized tensor trick homotopy' (see Definition 4.1).

We also prove the coalgebraic versions of Theorem A and Theorem B: Theorem 7.7 and Theorem 7.8. The following theorem is an application of these results.

Theorem C (Transfer Theorem for $\mathcal{O}_\infty$-algebras, Theorem 8.2)**.** *Suppose that the ground ring $\Bbbk$ contains $\mathbb{Q}$. Let $\mathcal{O}$ be any Koszul operad. Given a contraction $\mathscr{D}$ and an $\mathcal{O}_\infty$-algebra structure ρ on A, there is an $\mathcal{O}_\infty$-algebra structure ρ' on B and quasi-isomorphisms of $\mathcal{O}_\infty$-algebras* $\begin{tikzcd} (A,\rho) \arrow[r, shift left, "\sim"] & (B,\rho') \arrow[l, shift left] \end{tikzcd}$.

Corollary (Minimality theorem for $\mathcal{O}_\infty$-algebras, Corollary 8.3)**.** *Suppose that $\Bbbk$ is a field of characteristic zero. Let A be a chain complex, and suppose given an $\mathcal{O}_\infty$-algebra structure ρ on A. Then there is an $\mathcal{O}_\infty$-algebra structure ρ' on the homology $\mathrm{H}_*(A)$, with trivial differential, and quasi-isomorphisms of $\mathcal{O}_\infty$-algebras*

$$\begin{tikzcd} A \arrow[r, shift left, "\sim"] & \mathrm{H}_*(A) \arrow[l, shift left] \end{tikzcd}$$

Specialization to the particular Koszul operads $\mathcal{A}s$, $\mathcal{C}om$ and $\mathcal{L}ie$ gives in one stroke homological perturbation theory proofs of the transfer and minimality theorems for A_∞, C_∞ and L_∞-algebras.

Remark. Our definition of an $\mathcal{O}$-algebra contraction generalizes the classical definition of an algebra contraction [22, §2],[16, §2.2]. Indeed, if $\mathscr{D}$ is an algebra contraction in the classical sense then the formula

$$h_n = \sum_{i+1+j} 1^{\otimes i} \otimes h \otimes (gf)^{\otimes j}$$

exhibits $\mathscr{D}$ as an $\mathcal{O}$-algebra contraction in the sense of our definition, where $\mathcal{O}$ is the non-symmetric operad governing binary algebras. Conditions (1) and (2) are

consequences of the 'Tensor trick' [14, §3.2],[16, §3],[22, (2.2.0*)], and the reader is invited to check Condition (3) directly.

In fact, for non-symmetric operads $\mathcal{N}$, nothing prevents one from defining a contraction of $\mathcal{N}$-algebras to be a contraction $\mathscr{D}$ where f and g are morphisms of $\mathcal{N}$-algebras and where h is an '$\mathcal{N}$-algebra homotopy' in the sense that for all $\mu \in \mathcal{N}(n)$

$$h\mu_A = (-1)^{|\mu|}\mu_A h_n,$$

where h_n is given by the above formula. With this definition, Theorem A and Theorem B are valid (without any restriction on the ground ring in Theorem B), and are consequences of the classical Algebra Perturbation Lemma [22, (2.1*)].

However, as was pointed out in [16, Remark, end of §2.2], such a definition is not suited for symmetric operads, because the formula for h_n above is, except in trivial cases, not equivariant with respect to the action of the symmetric group Σ_n permuting the tensor factors of $A^{\otimes n}$. Condition (3) is a weakening of the above formula for h_n. Because we do not require explicit formulas for the higher homotopies h_n in terms of f, g and h, we are forced to make them part of the data and to impose Conditions (1) and (2). We would also like to remark that, as will be seen in Section 3, *Condition* (3) *is the essential property of algebra homotopies which makes the Algebra Perturbation Lemma work.*

Outline of the paper. In Section 1 we review the relevant background material on homological perturbation theory. In Section 2 we define *thick maps* and introduce some notations for handling thick maps. In Section 3 we use the language of thick maps to reformulate the notion of an algebra contraction in a way that allows for generalizations. Then we isolate the property of algebra homotopies that make the Algebra Perturbation Lemma work. Thick maps having this property are baptized *pseudo-derivations.* In Section 4 we show that the symmetrized tensor trick homotopy, although failing to be an algebra homotopy, is in fact a pseudo-derivation. In Section 5 we show how thick maps can be used to linearize Schur functors, and we generalize the classical 'Tensor Trick' to arbitrary Schur functors. In Section 6 we define *thick maps of $\mathcal{O}$-algebras* for operads $\mathcal{O}$ and show that there is a dg-category whose objects are $\mathcal{O}$-algebras and whose maps are thick maps of $\mathcal{O}$-algebras. We define contractions of $\mathcal{O}$-algebras and we prove Theorem A and Theorem B. Section 7 is dual to Section 6. In it we define thick maps of $\mathcal{C}$-coalgebras, where $\mathcal{C}$ is a cooperad, and we state the duals of Theorem A and Theorem B. Finally, in Section 8 we prove Theorem C and its corollary.

Conventions. In this paper, the term 'chain complex' means unbounded chain complex over a commutative ground ring $\Bbbk$. Differentials have degree -1. If A is a chain complex, then d_A will denote its differential. Recall that a *dg-category* is a category $\mathscr{A}$ enriched over chain complexes, i.e., the data of a collection of objects $\operatorname{Ob}\mathscr{A}$ and for every two objects A and B a chain complex $\operatorname{Hom}_{\mathscr{A}}(A, B)$, elements of which will be referred to as *maps* from A to B, together with natural composition and unit morphisms that satisfy the usual unit and associativity axioms, see for instance [24]. We will use ∂ as a generic notation for the differential in $\operatorname{Hom}_{\mathscr{A}}(A, B)$. Thus, in the dg-category $\mathscr{C}$ of chain complexes, $\partial(f) = d_B f - (-1)^{|f|} f d_A$ for maps $f \in \operatorname{Hom}_{\mathscr{C}}(A, B)$. The term *morphism* is reserved for maps f of degree 0 with $\partial(f) = 0$.

1. Background

In this section we will review the classical results of homological perturbation theory. The central notion, which goes back to Eilenberg and MacLane [8, §12], is that of a contraction.

Definition 1.1. A *contraction* is a diagram of maps of chain complexes

$$\mathscr{D}\colon \; \begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd},$$

where $|f| = |g| = 0$, $|h| = 1$, $\partial(f) = 0$, $\partial(g) = 0$, and

$$\partial(h) = gf - 1_A, \qquad fg = 1_B.$$

Furthermore, we impose the annihilation conditions

$$fh = 0, \quad hh = 0, \quad hg = 0.$$

We say that $\mathscr{D}$ is a *filtered contraction* if A and B are equipped with bounded below exhaustive filtrations which are preserved by the maps f, g and h.

In plain English, f and g are morphisms of chain complexes with $fg = 1_B$ and h is a chain homotopy from gf to 1_A. Thus, B is a strong deformation retract of A. For this reason, the term 'SDR-data' is often used as an alternative to 'contraction'. We have stated the definition in formulas rather than in words to make it clear that contractions make sense in any dg-category.

Remark 1.2. It is harmless to assume the annihilation conditions, as was pointed out in [27]. If they are not satisfied, then one can replace h by $h'' = -h'dh'$, where $h' = \partial(h)h\partial(h)$, to get a contraction.

A *perturbation* of A is a map $t\colon A \to A$ of degree -1 such that $\partial(t) + t^2 = 0$, or, equivalently, $(d_A + t)^2 = 0$. Let A^t denote the chain complex A with new differential $d_A + t$. The following result is the basis for the theory.

Theorem 1.3 (Basic Perturbation Lemma, [6],[13])**.** *If t is a perturbation of A such that $1 - ht$ is invertible then setting $\Sigma = t(1 - ht)^{-1}$ the following formulas define a perturbation t' of B and a new contraction*

$$\mathscr{D}^t\colon \; \begin{tikzcd} A^t \arrow[loop left, "h'"] \arrow[r, shift left, "f'"] & B^{t'} \arrow[l, shift left, "g'"] \end{tikzcd}, \qquad \begin{array}{ll} f' = f + f\Sigma h, & g' = g + h\Sigma g, \\ h' = h + h\Sigma h, & t' = f\Sigma g. \end{array}$$

Remark 1.4. In the original statement of the Basic Perturbation Lemma [13] one assumes that $\mathscr{D}$ is a filtered contraction and that the perturbation t lowers the filtration on A. Then the infinite series $\sum_{n\geq 0}(ht)^n$ converges pointwise and defines an inverse of $1 - ht$. It was observed in [2] that invertibility of $1 - ht$ is a sufficient hypothesis. Observe also that invertibility of $1 - ht$ is equivalent to invertibility of $1 - th$. Indeed, $(1 - th)^{-1} = 1 + t(1 - ht)^{-1}h$.

Definition 1.5 ([16],[22])**.** A contraction $\mathscr{D}$ is called a *contraction of algebras* if A and B are algebras, i.e., are equipped with morphisms of chain complexes $\mu_A\colon A \otimes A \to A$ and $\mu_B\colon B \otimes B \to B$, if f and g are morphisms of algebras, and if h is an *algebra homotopy*, which means that

$$h\mu_A = \mu_B(h \otimes gf + 1 \otimes h).$$

Theorem 1.6 (Algebra Perturbation Lemma, [16, §2.2], [22, (2.1*)])**.** *If $\mathscr{D}$ is a contraction of algebras and if the perturbation t is a derivation, i.e., $t\mu_A = \mu_A(t \otimes 1 + 1 \otimes t)$, then $\mathscr{D}^t$ is a contraction of algebras.*

The so called 'Tensor Trick' is a way of producing an algebra contraction starting from any contraction. Recall that the *tensor algebra* on a chain complex A is the chain complex

$$T(A) = \bigoplus_{n \geq 0} A^{\otimes n}$$

with multiplication $\mu\colon T(A) \otimes T(A) \to T(A)$ induced by the canonical isomorphisms $A^{\otimes p} \otimes A^{\otimes q} \cong A^{\otimes(p+q)}$.

Theorem 1.7 (Tensor Trick, [14, §3.2],[16, §3],[22, (2.2.0*)])**.** *For any contraction $\mathscr{D}$ the following is a contraction of algebras*

$$T(\mathscr{D})\colon \begin{tikzcd} T(A) \arrow[loop left, "H"] \arrow[r, shift left, "F"] & T(B) \arrow[l, shift left, "G"] \end{tikzcd},$$

where F, G and H act on tensors of length n by, respectively,

$$f^{\otimes n}, \quad g^{\otimes n}, \quad \sum_{i+1+j=n} 1^{\otimes i} \otimes h \otimes (gf)^{\otimes j}.$$

As remarked in [16, Remark, end of §2.2], if μ_A is a commutative operation, then the left hand side of the equation

$$h\mu_A = \mu_A(h \otimes gf + 1 \otimes h)$$

is symmetric but the right hand side is not. For this reason, the present notion of an algebra homotopy is not useful for commutative algebras or, more generally, for algebras where symmetries play a role. In what follows, we will look for a symmetric generalization of the notion of an algebra homotopy such that Theorem 1.6 and Theorem 1.7, appropriately modified, remain valid.

2. Thick maps

Definition 2.1. Let A and B be chain complexes. A *thick map* $\mathbf{f}\colon A \to B$ is a sequence of maps

$$\mathbf{f} = \{f_n\colon A^{\otimes n} \to B^{\otimes n}\}_{n \geq 0}$$

of the same degree $|\mathbf{f}|$. It is called *symmetric* if each f_n is equivariant with respect to the action of the symmetric group Σ_n permuting tensor factors.

There is a dg-category $T_{\mathbb{N}}(\mathscr{C})$ of thick maps. It has the same objects as the dg-category $\mathscr{C}$ of chain complexes but $\mathrm{Hom}_{T_{\mathbb{N}}(\mathscr{C})}(A, B)$ is the chain complex of thick maps from A to B. The $\Bbbk$-linear structure, differentials and compositions are defined by

$$\begin{aligned} (a\mathbf{f} + b\mathbf{f}')_n &= af_n + bf'_n, \\ \partial(\mathbf{f})_n &= d_{B^{\otimes n}} f_n - (-1)^{|f_n|} f_n d_{A^{\otimes n}}, \\ (\mathbf{g} \circ \mathbf{f})_n &= g_n \circ f_n, \end{aligned}$$

for $\mathbf{f}, \mathbf{f}'\colon A \to B$, $\mathbf{g}\colon B \to C$, $a, b \in \Bbbk$, and where $d_{A^{\otimes n}}$ is the usual tensor product differential on $A^{\otimes n}$. Chain complexes together with *symmetric* thick maps form a dg-subcategory $T_\Sigma(\mathscr{C})$ of $T_{\mathbb{N}}(\mathscr{C})$. We will now give names to thick map with special properties.

Definition 2.2. (1) A thick map $\mathbf{f}\colon A \to B$ is called a *morphism* if $f_n = f_1^{\otimes n}$ for all n.
(2) Let $\mathbf{l}$ and $\mathbf{r}$ be morphisms from A to B. A thick map $\mathbf{d}\colon A \to B$ is called an $(\mathbf{l}, \mathbf{r})$-*derivation* if $d_n = \sum_{i+1+j=n} l_1^{\otimes i} \otimes d_1 \otimes r_1^{\otimes j}$ for all n.
(3) For simplicity, a $(\mathbf{1}, \mathbf{1})$-derivation $\mathbf{d}\colon A \to A$ will be called a *derivation*.

Let us also introduce a notational device. If $\mathbf{f}\colon A \to B$ and $\mathbf{g}\colon C \to D$ are two thick maps, we can form the bi-indexed sequence

$$\mathbf{f} \otimes \mathbf{g} = \{f_p \otimes g_q\colon A^{\otimes p} \otimes C^{\otimes q} \to B^{\otimes p} \otimes D^{\otimes q}\}_{p,q \geq 0}.$$

One can also form the bi-indexed sequence $m^*(\mathbf{f})$ where

$$m^*(\mathbf{f}) = \{f_{p+q}\colon A^{\otimes p} \otimes A^{\otimes q} \to B^{\otimes p} \otimes B^{\otimes q}\}_{p,q \geq 0}.$$

Proposition 2.3. (1) *A thick map $\mathbf{f}\colon A \to B$ is a morphism if and only if*

$$m^*(\mathbf{f}) = \mathbf{f} \otimes \mathbf{f}.$$

(2) *Let $\mathbf{l}$ and $\mathbf{r}$ be morphisms from A to B. A thick map $\mathbf{d}\colon A \to B$ is an $(\mathbf{l}, \mathbf{r})$-derivation if and only if*

$$m^*(\mathbf{d}) = \mathbf{d} \otimes \mathbf{r} + \mathbf{l} \otimes \mathbf{d}.$$

Proof. By induction, the condition for $\mathbf{f}$ to be a morphism is equivalent to $f_{p+q} = f_p \otimes f_q$ for all p, q, and the condition for $\mathbf{d}$ to be an $(\mathbf{l}, \mathbf{r})$-derivation is equivalent to $d_{p+q} = d_p \otimes r_q + l_p \otimes d_q$ for all p, q. □

3. Thick contractions

Using thick maps we can reformulate the notion of an algebra contraction in a way that lends itself to generalizations. By a *thick contraction* we mean a contraction in the dg-category $T_{\mathbb{N}}(\mathscr{C})$.

Proposition 3.1. *Any contraction $\mathscr{D}$ has a unique extension to a thick contraction*

$$\underline{\mathscr{D}}\colon \begin{tikzcd} A \arrow[loop left, "\mathbf{h}"] \arrow[r, shift left, "\mathbf{f}"] & B \arrow[l, shift left, "\mathbf{g}"] \end{tikzcd}$$

where $\mathbf{f}$ and $\mathbf{g}$ are morphisms and $\mathbf{h}$ is a $(\mathbf{1}, \mathbf{gf})$-derivation. Furthermore, $\mathscr{D}$ is an algebra contraction if and only if $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are compatible with the algebraic structure on A and B in the sense that

$$f_1 \mu_A = \mu_B f_2, \quad g_1 \mu_B = \mu_A g_2, \quad h_1 \mu_A = \mu_A h_2.$$

Proof. By Proposition 2.3, requiring that $\mathbf{f}$, $\mathbf{g}$ are morphisms and that $\mathbf{h}$ is a $(\mathbf{1}, \mathbf{gf})$-derivation leaves us with no choice but to set

$$f_n = f^{\otimes n}, \quad g_n = g^{\otimes n}, \quad h_n = \sum_{i+1+j=n} 1^{\otimes i} \otimes h \otimes (gf)^{\otimes j}.$$

But these formulas coincide with the formulas in the Tensor Trick (Theorem 1.7), and it is a consequence of that theorem that they define a thick contraction. Next, $\mathscr{D}$ is an algebra contraction (Definition 1.5) if and only if

$$f m_A = m_B f^{\otimes 2}, \quad g \mu_B = \mu_A g^{\otimes 2}, \quad h \mu_A = \mu_A (h \otimes gf + 1 \otimes h).$$

In view of our definition of $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, these conditions are the same as the conditions in the statement of the proposition. □

We repeat that the problem with algebra homotopies is the asymmetry in the expression $h \otimes gf + 1 \otimes h$. In other words, the problem is that if a thick map $\mathbf{h}$ is a $(\mathbf{1}, \mathbf{gf})$-derivation, then it can hardly be symmetric in the sense of Definition 2.1. The goal for the remainder of this section is the following: *Generalize the condition '$\mathbf{h}$ is a $(\mathbf{1}, \mathbf{gf})$-derivation' to a condition that makes sense for symmetric thick maps.* There are two constraints:

- The condition should be sufficiently close to the $(\mathbf{1}, \mathbf{gf})$-derivation condition so that the proof of the Algebra Perturbation Lemma goes through.
- The condition should be flexible enough so as to allow for a 'symmetric tensor trick', i.e., an extension of any contraction to a *symmetric* thick contraction which satisfies the condition.

We will argue that the following definition contains the solution to this problem.

Definition 3.2. A thick map $\mathbf{h} \colon A \to A$ is called a *pseudo-derivation* if

$$(\mathbf{h} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{h}) m^*(\mathbf{h}) = -m^*(\mathbf{h})(\mathbf{h} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{h}) = \mathbf{h} \otimes \mathbf{h}.$$

In other words, $\mathbf{h}$ is a pseudo-derivation if for all $p, q \geq 0$

$$(h_p \otimes 1 - 1 \otimes h_q) h_{p+q} = -h_{p+q}(h_p \otimes 1 - 1 \otimes h_q) = h_p \otimes h_q.$$

For the rest of the section, fix a thick contraction

$$\underline{\mathscr{D}} \colon \ \mathbf{h} \circlearrowright A \underset{\mathbf{g}}{\overset{\mathbf{f}}{\rightleftarrows}} B \ .$$

To begin with, let us note that pseudo-derivations generalize $(\mathbf{1}, \mathbf{gf})$-derivations.

Proposition 3.3. *If the homotopy $\mathbf{h}$ in $\underline{\mathscr{D}}$ is a $(\mathbf{1}, \mathbf{gf})$-derivation then $\mathbf{h}$ is a pseudo-derivation.*

Proof. This is a simple calculation:

$$\begin{aligned}(\mathbf{h} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{h}) m^*(\mathbf{h}) &= (\mathbf{h} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{h})(\mathbf{1} \otimes \mathbf{h} + \mathbf{h} \otimes \mathbf{gf}) \\ &= \mathbf{h} \otimes \mathbf{h} - \mathbf{1} \otimes \mathbf{hh} + \mathbf{hh} \otimes \mathbf{gf} + \mathbf{h} \otimes \mathbf{hgf} \\ &= \mathbf{h} \otimes \mathbf{h}.\end{aligned}$$

Here we have used the annihilation conditions $\mathbf{hh} = 0$ and $\mathbf{hg} = 0$. Similarly, one verifies that $-m^*(\mathbf{h})(\mathbf{h} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{h}) = \mathbf{h} \otimes \mathbf{h}$. $\square$

Fix a thick perturbation $\mathbf{t}$ of A, i.e., a thick map of degree -1 satisfying $\partial(\mathbf{t}) + \mathbf{t}^2 = 0$. Assume that $\mathbf{1} - \mathbf{ht}$ (or equivalently $\mathbf{1} - \mathbf{th}$) is invertible, so that we can use the formulas of the Basic Perturbation Lemma (Theorem 1.3) to define thick maps $\mathbf{f}'$, $\mathbf{g}'$, $\mathbf{h}'$, $\mathbf{t}'$. The following theorem, which shows that the pseudo-derivation property is sufficient to make the Algebra Perturbation Lemma work, is the main result of this section.

Theorem 3.4. *Let $\underline{\mathscr{D}}$ be a thick contraction. If $\mathbf{f}$ and $\mathbf{g}$ are morphisms, $\mathbf{h}$ is a pseudo-derivation and $\mathbf{t}$ is a derivation then $\mathbf{f}'$ and $\mathbf{g}'$ are morphisms, $\mathbf{h}'$ is a pseudo-derivation, $\mathbf{t}'$ is a derivation, $t = t_1$ and $t' = t'_1$ are perturbations of A and B, respectively, and*

$$\underline{\mathscr{D}}^{\mathbf{t}} \colon \ \mathbf{h}' \circlearrowright A^t \underset{\mathbf{g}'}{\overset{\mathbf{f}'}{\rightleftarrows}} B^{t'}$$

is a thick contraction. Furthermore, if $\mathbf{h}$ is symmetric, then so is $\mathbf{h}'$.

The proof of this theorem will occupy the rest of the section.

Proposition 3.5. *If* $\mathbf{h}$ *is a pseudo-derivation and* $\mathbf{t}$ *is a derivation then* $\mathbf{h}'$ *is a pseudo-derivation.*

Proof. We need to show that $(\mathbf{h}'\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h}')m^*(\mathbf{h}') = -m^*(\mathbf{h}')(\mathbf{h}'\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h}') = \mathbf{h}'\otimes\mathbf{h}'$. If we multiply the right hand side from the left with $(\mathbf{1}-\mathbf{ht})\otimes(\mathbf{1}-\mathbf{ht})$ and from the right with $m^*(\mathbf{1}-\mathbf{th})$ and use that $(\mathbf{1}-\mathbf{ht})\mathbf{h}' = \mathbf{h}'(\mathbf{1}-\mathbf{th}) = \mathbf{h}$ we get

$$\begin{aligned}
\big((\mathbf{1}-\mathbf{ht})\otimes(\mathbf{1}-\mathbf{ht})\big)&(\mathbf{h}'\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h}')m^*(\mathbf{h}')m^*(\mathbf{1}-\mathbf{th}) \\
&= (\mathbf{h}\otimes(\mathbf{1}-\mathbf{ht})-(\mathbf{1}-\mathbf{ht})\otimes\mathbf{h})m^*(\mathbf{h}) \\
&= (\mathbf{h}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h})m^*(\mathbf{h}) - (\mathbf{h}\otimes\mathbf{h})(\mathbf{t}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{t})m^*(\mathbf{h}) \\
&= \mathbf{h}\otimes\mathbf{h} - (\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{th}) \\
&= (\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{1}-\mathbf{th}) \\
&= \big((\mathbf{1}-\mathbf{ht})\otimes(\mathbf{1}-\mathbf{ht})\big)(\mathbf{h}'\otimes\mathbf{h}')m^*(\mathbf{1}-\mathbf{th}).
\end{aligned}$$

Since $(\mathbf{1}-\mathbf{ht})$ and $(\mathbf{1}-\mathbf{th})$ are invertible, the above equation implies that

$$(\mathbf{h}'\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h}')m^*(\mathbf{h}') = \mathbf{h}'\otimes\mathbf{h}'.$$

Similarly one verifies that $-m^*(\mathbf{h}')(\mathbf{h}'\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h}') = \mathbf{h}'\otimes\mathbf{h}'$. □

We will see in Proposition 3.8 below that the hypotheses in Theorem 3.4 imply the following additional conditions:

Module conditions.

$$\begin{aligned}
(\mathbf{f}\otimes\mathbf{1})m^*(\mathbf{h}) &= \mathbf{f}\otimes\mathbf{h} \qquad & m^*(\mathbf{h})(\mathbf{g}\otimes\mathbf{1}) &= \mathbf{g}\otimes\mathbf{h} \\
(\mathbf{1}\otimes\mathbf{f})m^*(\mathbf{h}) &= \mathbf{h}\otimes\mathbf{f} \qquad & m^*(\mathbf{h})(\mathbf{1}\otimes\mathbf{g}) &= \mathbf{h}\otimes\mathbf{g}
\end{aligned}$$

The module conditions together with the pseudo-derivation property are exactly what you need to ensure that $\mathbf{f}'$ and $\mathbf{g}'$ are morphisms and that $\mathbf{t}'$ is a derivation provided that $\mathbf{f}$ and $\mathbf{g}$ are morphisms and $\mathbf{t}$ is a derivation.

Proposition 3.6. *Suppose that* $\mathbf{h}$ *is a pseudo-derivation, that the module conditions are satisfied and that* $\mathbf{t}$ *is a derivation.*

(1) *If* $\mathbf{f}$ *is a morphism then so is* $\mathbf{f}'$*.*
(2) *If* $\mathbf{g}$ *is a morphism then so is* $\mathbf{g}'$*.*
(3) *If* $\mathbf{f}$ *and* $\mathbf{g}$ *are morphisms then* $\mathbf{t}'$ *is a derivation.*

Proof. (1): We need to verify that $m^*(\mathbf{f}') = \mathbf{f}'\otimes\mathbf{f}'$ under the assumption $m^*(\mathbf{f}) = \mathbf{f}\otimes\mathbf{f}$. Observe that $\mathbf{f}' = \mathbf{f}+\mathbf{f}'\mathbf{th}$. Therefore,

$$\begin{aligned}
(\mathbf{f}'\otimes\mathbf{f}')m^*(\mathbf{th}) &= (\mathbf{f}'\otimes\mathbf{f}')(\mathbf{t}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{t})m^*(\mathbf{h}) \\
&= (\mathbf{f}'\mathbf{t}\otimes(\mathbf{f}+\mathbf{f}'\mathbf{th})+(\mathbf{f}+\mathbf{f}'\mathbf{th})\otimes\mathbf{f}'\mathbf{t})m^*(\mathbf{h}) \\
&= (\mathbf{f}'\mathbf{t}\otimes\mathbf{1})(\mathbf{1}\otimes\mathbf{f})m^*(\mathbf{h}) + (\mathbf{1}\otimes\mathbf{f}'\mathbf{t})(\mathbf{f}\otimes\mathbf{1})m^*(\mathbf{h}) \\
&\quad - (\mathbf{f}'\mathbf{t}\otimes\mathbf{f}'\mathbf{t})(\mathbf{h}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h})m^*(\mathbf{h}) \\
&= (\mathbf{f}'\mathbf{t}\otimes\mathbf{1})(\mathbf{h}\otimes\mathbf{f}) + (\mathbf{1}\otimes\mathbf{f}'\mathbf{t})(\mathbf{f}\otimes\mathbf{h}) - (\mathbf{f}'\mathbf{t}\otimes\mathbf{f}'\mathbf{t})(\mathbf{h}\otimes\mathbf{h}) \\
&= \mathbf{f}'\mathbf{th}\otimes\mathbf{f} + \mathbf{f}\otimes\mathbf{f}'\mathbf{th} + \mathbf{f}'\mathbf{th}\otimes\mathbf{f}'\mathbf{th} \\
&= (\mathbf{f}'-\mathbf{f})\otimes\mathbf{f} + \mathbf{f}\otimes(\mathbf{f}'-\mathbf{f}) + (\mathbf{f}'-\mathbf{f})\otimes(\mathbf{f}'-\mathbf{f}) \\
&= \mathbf{f}'\otimes\mathbf{f}' - \mathbf{f}\otimes\mathbf{f}.
\end{aligned}$$

Here we have used that $\mathbf{h}$ is a pseudo-derivation, that $\mathbf{t}$ is a derivation and the module conditions involving $\mathbf{f}$. The above gives that

$$(\mathbf{f}'\otimes\mathbf{f}')m^*(\mathbf{1}-\mathbf{th}) = \mathbf{f}\otimes\mathbf{f} = m^*(\mathbf{f}) = m^*(\mathbf{f}'(\mathbf{1}-\mathbf{th})) = m^*(\mathbf{f}')m^*(\mathbf{1}-\mathbf{th}),$$

and this implies that $\mathbf{f}'\otimes\mathbf{f}' = m^*(\mathbf{f}')$ since $\mathbf{1}-\mathbf{th}$ is invertible.

(2): This is proved as (1) but uses the module conditions involving $\mathbf{g}$ instead.

(3): Note that $\mathbf{t}' = \mathbf{f}'\mathbf{tg}$. Since $\mathbf{hg} = 0$ and $\mathbf{f}'(\mathbf{1}-\mathbf{th}) = \mathbf{f}$, we have that $\mathbf{f}'\mathbf{g} = \mathbf{f}'(\mathbf{1}-\mathbf{th})\mathbf{g} = \mathbf{fg} = \mathbf{1}$. By (1), $\mathbf{f}'$ is a morphisms. Combining these facts we get that

$$\begin{aligned} m^*(\mathbf{t}') &= m^*(\mathbf{f}')m^*(\mathbf{t})m^*(\mathbf{g}) = (\mathbf{f}'\otimes\mathbf{f}')(\mathbf{t}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{t})(\mathbf{g}\otimes\mathbf{g}) \\ &= \mathbf{f}'\mathbf{tg}\otimes\mathbf{f}'\mathbf{g} + \mathbf{f}'\mathbf{g}\otimes\mathbf{f}'\mathbf{tg} = \mathbf{t}'\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{t}', \end{aligned}$$

so $\mathbf{t}'$ is indeed a derivation. □

To show that the module conditions are satisfied under the hypotheses of Theorem 3.4, we will introduce an auxiliary set of conditions on $\underline{\mathscr{D}}$, called the 'annihilation conditions', summarized as follows: all possible ways of forming maps $m^*(\mathbf{x})(\mathbf{y}\otimes\mathbf{z})$ or $(\mathbf{x}\otimes\mathbf{y})m^*(\mathbf{z})$ where $\{\mathbf{h}\}\subseteq\{\mathbf{x},\mathbf{y},\mathbf{z}\}\subseteq\{\mathbf{f},\mathbf{g},\mathbf{h}\}$ should yield the zero map.

Annihilation conditions.

$$\begin{aligned} (\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{h}) &= 0, & m^*(\mathbf{h})(\mathbf{h}\otimes\mathbf{h}) &= 0,\\ (\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{g}) &= 0, & m^*(\mathbf{f})(\mathbf{h}\otimes\mathbf{h}) &= 0,\\ (\mathbf{f}\otimes\mathbf{h})m^*(\mathbf{h}) &= 0, & m^*(\mathbf{h})(\mathbf{g}\otimes\mathbf{h}) &= 0,\\ (\mathbf{h}\otimes\mathbf{f})m^*(\mathbf{h}) &= 0, & m^*(\mathbf{h})(\mathbf{h}\otimes\mathbf{g}) &= 0,\\ (\mathbf{f}\otimes\mathbf{f})m^*(\mathbf{h}) &= 0, & m^*(\mathbf{h})(\mathbf{g}\otimes\mathbf{g}) &= 0,\\ (\mathbf{f}\otimes\mathbf{h})m^*(\mathbf{g}) &= 0, & m^*(\mathbf{f})(\mathbf{g}\otimes\mathbf{h}) &= 0,\\ (\mathbf{h}\otimes\mathbf{f})m^*(\mathbf{g}) &= 0, & m^*(\mathbf{f})(\mathbf{h}\otimes\mathbf{g}) &= 0. \end{aligned}$$

The annihilation conditions, albeit outnumbering the module conditions, are easier to verify, and, getting ahead of ourselves, we will take advantage of this in proving Theorem 4.4.

Proposition 3.7. (1) *The homotopy* $\mathbf{h}$ *is a pseudo-derivation if and only if the annihilation conditions in the four first rows are satisfied.*

(2) *The module conditions are equivalent to the annihilation conditions in the five last rows.*

Proof. (1): Consider the differential of the map $(\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{h})$:

$$\begin{aligned} &\partial((\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{h}))\\ &\quad = ((\mathbf{gf}-\mathbf{1})\otimes\mathbf{h})m^*(\mathbf{h}) - (\mathbf{h}\otimes(\mathbf{gf}-\mathbf{1}))m^*(\mathbf{h}) + (\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{gf}-\mathbf{1})\\ &\quad = (\mathbf{h}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h})m^*(\mathbf{h}) - \mathbf{h}\otimes\mathbf{h}\\ &\qquad + (\mathbf{g}\otimes\mathbf{1})(\mathbf{f}\otimes\mathbf{h})m^*(\mathbf{h}) - (\mathbf{1}\otimes\mathbf{g})(\mathbf{h}\otimes\mathbf{f})m^*(\mathbf{h}) - (\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{g})m^*(\mathbf{f}). \end{aligned}$$

From this expression, one sees that the equality $(\mathbf{h}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h})m^*(\mathbf{h}) = \mathbf{h}\otimes\mathbf{h}$ follows from the first four annihilation conditions in the left column. Conversely, these four annihilation conditions follow from the equality $(\mathbf{h}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h})m^*(\mathbf{h}) = \mathbf{h}\otimes\mathbf{h}$:

$$(\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{h}) = (\mathbf{h}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h})m^*(\mathbf{h})m^*(\mathbf{h}) = (\mathbf{h}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h})m^*(\mathbf{hh}) = 0,$$

and similarly $(\mathbf{h}\otimes\mathbf{h})m^*(\mathbf{g})=0$. Next,

$$\begin{aligned}(\mathbf{f}\otimes\mathbf{h})m^*(\mathbf{h}) &= (\mathbf{f}\otimes\mathbf{1})(\mathbf{1}\otimes\mathbf{h})m^*(\mathbf{h}) = (\mathbf{f}\otimes\mathbf{1})((\mathbf{h}\otimes\mathbf{1})m^*(\mathbf{h})-\mathbf{h}\otimes\mathbf{h})\\ &= (\mathbf{fh}\otimes\mathbf{1})m^*(\mathbf{h})-\mathbf{fh}\otimes\mathbf{h}=0,\end{aligned}$$

and similarly $(\mathbf{h}\otimes\mathbf{f})m^*(\mathbf{h})=0$. Likewise, the condition $-m^*(\mathbf{h})(\mathbf{h}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{h})=\mathbf{h}\otimes\mathbf{h}$ is equivalent to the first four annihilation conditions in the right column.

(2): By the same token, each individual module condition is equivalent to three annihilation conditions. The module condition $(\mathbf{f}\otimes\mathbf{1})m^*(\mathbf{h})=\mathbf{f}\otimes\mathbf{h}$ is equivalent to the three annihilation conditions

$$(\mathbf{f}\otimes\mathbf{h})m^*(\mathbf{h})=0,\quad (\mathbf{f}\otimes\mathbf{h})m^*(\mathbf{g})=0,\quad (\mathbf{f}\otimes\mathbf{f})m^*(\mathbf{h})=0.$$

The proof is similar to the proof of (1) and is left to the reader. One direction is seen by differentiating the expression $(\mathbf{f}\otimes\mathbf{h})m^*(\mathbf{h})$. After doing the same thing for each module condition, one sees that they are collectively equivalent to the annihilation conditions in the five last rows. □

As promised, we can now prove the following:

Proposition 3.8. *If* $\mathbf{f}$ *and* $\mathbf{g}$ *are morphisms and* $\mathbf{h}$ *is a pseudo-derivation then all annihilation conditions are satisfied, and hence the module conditions are automatically satisfied.*

Proof. By Proposition 3.7 (1), if $\mathbf{h}$ is a pseudo-derivation then the annihilation conditions in the four first rows are satisfied. If $\mathbf{f}$ and $\mathbf{g}$ are morphisms, then the annihilation conditions in the three remaining rows follow from the conditions $\mathbf{fh}=0$ and $\mathbf{hg}=0$:

$$(\mathbf{f}\otimes\mathbf{f})m^*(\mathbf{h})=m^*(\mathbf{f})m^*(\mathbf{h})=m^*(\mathbf{fh})=0,$$

$$(\mathbf{f}\otimes\mathbf{h})m^*(\mathbf{g})=(\mathbf{f}\otimes\mathbf{h})(\mathbf{g}\otimes\mathbf{g})=\mathbf{fg}\otimes\mathbf{hg}=0,$$

and so on. That the module conditions hold then follows from Proposition 3.7 (2). □

Proof of Theorem 3.4. By Proposition 3.8, the module conditions are satisfied, so by Proposition 3.6, $\mathbf{f}'$ and $\mathbf{g}'$ are morphisms, $\mathbf{h}'$ is a pseudo-derivation and $\mathbf{t}'$ is a derivation. We need to show that $t=t_1$ and $t'=t'_1$ are perturbations of A and B, respectively, and that $\underline{\mathscr{D}}^{\mathbf{t}}$ is a thick contraction. The n^{th} level of the diagram $\underline{\mathscr{D}}^{\mathbf{t}}$ is equal to the diagram

$$\mathscr{D}_n^{t_n}:\ h'_n \circlearrowright (A^{\otimes n})^{t_n} \underset{g'_n}{\overset{f'_n}{\rightleftarrows}} (B^{\otimes n})^{t'_n}$$

obtained by perturbing the n^{th} level $\mathscr{D}_n$ of the thick contraction $\underline{\mathscr{D}}$ using the perturbation t_n of $A^{\otimes n}$. By the Basic Perturbation Lemma, t'_n is a perturbation of $B^{\otimes n}$ and $\mathscr{D}_n^{t_n}$ is a contraction. In particular, t and t' are perturbations of A and B, respectively. Furthermore, the relations $\mathbf{f}'\mathbf{g}'=\mathbf{1}$, $\mathbf{f}'\mathbf{h}'=0$, $\mathbf{h}'\mathbf{h}'=0$ and $\mathbf{h}'\mathbf{g}'=0$ hold because they do so levelwise. However, to verify that $\underline{\mathscr{D}}^{\mathbf{t}}$ is a thick contraction, it is not enough to know that each individual level is a contraction, we will also need the fact that $\mathbf{t}$ and $\mathbf{t}'$ are derivations. Observe that

$$\partial(\mathbf{h}')_n = d_{(A^t)^{\otimes n}}h'_n + h'_n d_{(A^t)^{\otimes n}}$$

Since $\mathbf{t}$ is a derivation, the tensor product differential $d_{(A^t)^{\otimes n}}$ in $(A^t)^{\otimes n}$ coincides with the perturbed differential $d_{A^{\otimes n}}+t_n$ of $(A^{\otimes n})^{t_n}$. Since each $\mathscr{D}_n^{t_n}$ is a contraction, this implies that

$$\partial(\mathbf{h}') = \mathbf{g}'\mathbf{f}' - \mathbf{1}.$$

Similarly, using that also $\mathbf{t}'$ is a derivation one verifies that $\partial(\mathbf{f}') = 0$ and that $\partial(\mathbf{g}') = 0$. This finishes the proof. □

Note that the condition to be a pseudo-derivation makes sense for symmetric thick maps. We will now turn to the symmetric tensor trick. We will return to the question of compatibility with algebraic structure in Section 6.

Remark 3.9. The reason for the name 'module conditions' is the following: Suppose that A and B are associative algebras and that $g_1\colon B \to A$ is a morphism of algebras. Then A can be viewed as a left B-module via $\mu_A(g_1 \otimes 1)\colon B \otimes A \to A$. Suppose moreover that $\mu_A h_2 = h_1 \mu_A$. Then the module condition $h_2(g_1 \otimes 1) = g_1 \otimes h_1$ implies that h_1 is a morphism of B-modules (of degree 1). The other module conditions have similar interpretations.

4. Symmetric tensor trick

By Proposition 3.1 any contraction $\mathscr{D}$ can be extended to a thick contraction $\underline{\mathscr{D}}$ where $\mathbf{f}$ and $\mathbf{g}$ are morphisms and $\mathbf{h}$ is a $(\mathbf{1}, \mathbf{gf})$-derivation. In this section we will symmetrize $\mathbf{h}$ to obtain a *symmetric* thick contraction $\underline{\mathscr{D}}^{\Sigma}$ which extends $\mathscr{D}$. The symmetrized homotopy $\mathbf{h}^{\Sigma}$ is no longer a $(\mathbf{1}, \mathbf{gf})$-derivation, but we will show that it is a pseudo-derivation. Throughout this section we will assume that $\mathbb{Q} \subseteq \Bbbk$. This assumption is necessary, see Proposition 5.6.

Fix a contraction $\mathscr{D}$, and consider its extension to a thick contraction $\underline{\mathscr{D}}$ given by Proposition 3.1:

$$f_n = f^{\otimes n}, \quad g_n = g^{\otimes n}, \quad h_n = \sum_{i+1+j=n} 1^{\otimes i} \otimes h \otimes \pi^{\otimes j}.$$

Here $\pi = gf$. Evidently, the thick maps $\mathbf{f}$ and $\mathbf{g}$ are symmetric, but $\mathbf{h}$ is not.

Definition 4.1. The *symmetrized tensor trick homotopy* $\mathbf{h}^{\Sigma}\colon A \to A$ is the thick map defined by

$$h_n^{\Sigma_n} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} h_n^{\sigma},$$

where $h_n^{\sigma} = \sigma^{-1} h_n \sigma$.

The idea of symmetrizing the tensor trick homotopy appears in [15],[23],[18],[19] and presumably in many other places, but the author is not aware of any written source where the formal properties of the symmetrized homotopy are worked out in detail. In particular, we believe that the discovery that $\mathbf{h}^{\Sigma}$ is a pseudo-derivation is new, see Theorem 4.4 below.

Proposition 4.2. *The symmetrized homotopy* $\mathbf{h}^{\Sigma}\colon A \to A$ *can be decomposed as*

$$\mathbf{h}^{\Sigma} = \mathbf{q}\mathbf{h}^{der} = \mathbf{h}^{der}\mathbf{q}$$

where $\mathbf{h}^{der}$ *and* $\mathbf{q}$ *are the symmetric thick maps from* A *to itself given by*

$$h_n^{der} = \sum_{i+1+j=n} 1^{\otimes i} \otimes h \otimes 1^{\otimes j}$$

and

$$q_n = \sum_{\epsilon \in \{0,1\}^n} Q^n_{|\epsilon|} \pi^{\epsilon_1} \otimes \ldots \otimes \pi^{\epsilon_n}.$$

Here, $\pi = gf$*,* $|\epsilon| = \epsilon_1 + \ldots + \epsilon_n$*, and*

$$Q^n_k = \frac{k!(n-1-k)!}{n!}$$

if $k < n$*. We define* $Q^n_n = 0$*.*

Proof. The n^{th} component of the symmetrized homotopy is given by the formula

$$h_n^{\Sigma_n} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} h_n^\sigma,$$

where $h_n^\sigma = \sigma^{-1} h_n \sigma$. Every $\sigma \in \Sigma_n$ determines a total order $<_\sigma$ of $\{1, \ldots, n\}$ by

$$i <_\sigma j \Longleftrightarrow \sigma(i) < \sigma(j).$$

We have that

$$h_n^\sigma = \sum_{j=1}^n \alpha_1 \otimes \ldots \otimes \alpha_{j-1} \otimes h \otimes \alpha_{j+1} \otimes \ldots \otimes \alpha_n,$$

where, for $i \neq j$,

$$\alpha_i = \begin{cases} 1 & \text{if } i <_\sigma j \\ \pi & \text{if } j <_\sigma i \end{cases}$$

Therefore, the sum of all h_n^σ is a linear combination of terms of the form

$$\pi^{\epsilon_1} \otimes \ldots \otimes \pi^{\epsilon_{j-1}} \otimes h \otimes \pi^{\epsilon_{j+1}} \otimes \ldots \otimes \pi^{\epsilon_n},$$

where $\epsilon_i \in \{0,1\}$. The coefficient of such a term is the number of total orders on the set $\{1, \ldots, n\}$ with the property that j is the j^{th} element and all elements of the set $\{i \mid \epsilon_i = 0\}$ precedes all elements of the set $\{i \mid \epsilon_i = 1\}$. The number of such orders is $k!(n-1-k)!$, where $k = |\{i \mid \epsilon_i = 1\}| = |\epsilon|$. Hence,

$$h_n^{\Sigma_n} = \sum_{j=1}^n \sum_{\substack{\epsilon \in \{0,1\}^n \\ \epsilon_j = 0}} Q^n_{|\epsilon|} \pi^{\epsilon_1} \otimes \ldots \otimes \pi^{\epsilon_{j-1}} \otimes h \otimes \pi^{\epsilon_{j+1}} \otimes \ldots \otimes \pi^{\epsilon_n}.$$

Since $h\pi = \pi h = 0$, this may be written as $\mathbf{h}^\Sigma = \mathbf{h}^{der}\mathbf{q} = \mathbf{q}\mathbf{h}^{der}$, as claimed. □

Remark 4.3. Observe that since $h\pi = \pi h = 0$ it does not matter how Q^n_n is defined, but we define it to be zero for definiteness.

Theorem 4.4. *The diagram*

$$\underline{\mathscr{D}}^\Sigma\colon\ \mathbf{h}^\Sigma \circlearrowright A \underset{\mathbf{g}}{\overset{\mathbf{f}}{\rightleftarrows}} B$$

is a symmetric thick contraction which extends $\mathscr{D}$*. Furthermore,* **f** *and* **g** *are morphisms and* $\mathbf{h}^\Sigma$ *is a pseudo-derivation.*

Proof. The relation $\partial(\mathbf{h}^\Sigma) = \mathbf{gf} - \mathbf{1}$ follows from the relation $\partial(\mathbf{h}) = \mathbf{gf} - \mathbf{1}$ because symmetrization is a morphism of chain complexes

$$\mathrm{Hom}_{\mathscr{C}}(A^{\otimes n}, A^{\otimes n}) \to \mathrm{Hom}_{\mathscr{C}}(A^{\otimes n}, A^{\otimes n})^{\Sigma_n}.$$

and because the thick map $\mathbf{gf} - \mathbf{1}$ is symmetric. The relation $\mathbf{fg} = \mathbf{1}$ is clear. By Proposition 4.2 we have $\mathbf{h}^\Sigma = \mathbf{q}\mathbf{h}^{der} = \mathbf{h}^{der}\mathbf{q}$. Since $fh = 0$, $hg = 0$ and $hh = 0$, it

follows that $\mathbf{f}\mathbf{h}^{der} = 0$, $\mathbf{h}^{der}\mathbf{g} = 0$ and $\mathbf{h}^{der}\mathbf{h}^{der} = 0$. Therefore, $\mathbf{f}\mathbf{h}^{\Sigma} = \mathbf{f}\mathbf{h}^{der}\mathbf{q} = 0$, $\mathbf{h}^{\Sigma}\mathbf{g} = \mathbf{q}\mathbf{h}^{der}\mathbf{g} = 0$ and $\mathbf{h}^{\Sigma}\mathbf{h}^{\Sigma} = \mathbf{q}\mathbf{h}^{der}\mathbf{h}^{der}\mathbf{q} = 0$. We have thus verified that $\underline{\mathscr{D}}^{\Sigma}$ is a contraction.

The maps $\mathbf{f}$ and $\mathbf{g}$ are by definition the morphisms that extend f and g. To prove that $\mathbf{h}^{\Sigma}$ is a pseudo-derivation, it suffices by Proposition 3.7 to verify the annihilation conditions. To do this, use the decomposition $\mathbf{h}^{\Sigma} = \mathbf{q}\mathbf{h}^{der} = \mathbf{h}^{der}\mathbf{q}$ and the fact that $\mathbf{h}^{der}$ is a derivation that annihilates $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}^{der}$. For instance,

$$\begin{aligned}(\mathbf{f}\otimes\mathbf{h}^{\Sigma})m^*(\mathbf{h}^{\Sigma}) &= (\mathbf{f}\otimes\mathbf{q}\mathbf{h}^{der})m^*(\mathbf{h}^{der})m^*(\mathbf{q})\\ &= (\mathbf{f}\otimes\mathbf{q}\mathbf{h}^{der})(\mathbf{h}^{der}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{h}^{der})m^*(\mathbf{q})\\ &= (-\mathbf{f}\mathbf{h}^{der}\otimes\mathbf{q}\mathbf{h}^{der}+\mathbf{f}\otimes\mathbf{q}\mathbf{h}^{der}\mathbf{h}^{der})m^*(\mathbf{q})\\ &= 0\end{aligned}$$

The other annihilation conditions are verified in a similar manner. □

Remark 4.5. We have proved that $\mathbf{h}^{\Sigma}$ is a pseudo-derivation and that $\underline{\mathscr{D}}^{\Sigma}$ satisfies the module conditions via Proposition 3.7 by verifying the annihilation conditions. The module conditions can also be verified directly. These verifications boil down to statements about the coefficients Q_k^n. For example, in proving that

$$(\mathbf{f}\otimes\mathbf{1})m^*(\mathbf{h}^{\Sigma}) = \mathbf{f}\otimes\mathbf{h}^{\Sigma},$$

one comes across the statement that the equality

$$\sum_{j=0}^{r}\binom{r}{j}Q_{j+k}^{n} = Q_k^{n-r}$$

holds for all non-negative integers r, k, n with $r + k < n$. Verifying directly that $\mathbf{h}^{\Sigma}$ is a pseudo-derivation involves a similar but more complicated equality. It is quite interesting that these combinatorial equalities are consequences of Proposition 3.7.

5. Tensor trick for Schur functors

Using the results of the previous sections, we will now generalize the Tensor Trick (Theorem 1.7) from the tensor algebra functor to arbitrary Schur functors. This is achieved by extending the functoriality of Schur functors to the dg-category of symmetric thick maps $T_{\Sigma}(\mathscr{C})$. In the next section we will see how the algebraic structure on $\mathcal{O}[A]$ comes into play when $\mathcal{O}$ is an operad or a cooperad.

Recall that a *symmetric sequence* is a collection $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ where $\mathcal{O}(n)$ is a chain complex with an action of the symmetric group Σ_n.

Definition 5.1 (cf. [9, §2.1.1])**.** The *Schur functor* associated to a symmetric sequence $\mathcal{O}$ is the functor $\mathcal{O}[-]\colon \mathscr{C}\to\mathscr{C}$ given on objects A by

$$\mathcal{O}[A] = \bigoplus_{n\geq 0}\mathcal{O}(n)\otimes_{\Sigma_n}A^{\otimes n},$$

and on morphisms $f\colon A\to B$ by

$$\mathcal{O}[f] = \bigoplus_{n\geq 0} 1\otimes_{\Sigma_n} f^{\otimes n}\colon \mathcal{O}[A]\to\mathcal{O}[B].$$

Note that the tensor algebra $T(A)$ is the value at A of the Schur functor associated to the symmetric sequence $\Bbbk\Sigma = \{\Bbbk\Sigma_n\}_{n\geq 0}$ of regular representations. In general, the Schur functor $\mathcal{O}[-]$ is non-additive. We will extend the Schur functor to the dg-category of symmetric thick maps $T_\Sigma(\mathscr{C})$. This extended Schur functor will be additive.

Proposition 5.2. *Any symmetric sequence $\mathcal{O}$ determines a dg-functor*

$$\mathcal{O}[-]\colon T_\Sigma(\mathscr{C}) \to \mathscr{C}$$

which coincides with the Schur functor $A \mapsto \mathcal{O}[A]$ on objects.

Proof. If $\mathbf{f}\colon A \to B$ is a symmetric thick map then let

$$\mathcal{O}[\mathbf{f}] = \bigoplus_{n\geq 0} 1 \otimes_{\Sigma_n} f_n \colon \mathcal{O}[A] \to \mathcal{O}[B].$$

It is straightforward to check that $\mathcal{O}[-]$ is $\Bbbk$-linear, that $\mathcal{O}[\partial(\mathbf{f})] = \partial(\mathcal{O}[\mathbf{f}])$ and that $\mathcal{O}[\mathbf{f} \circ \mathbf{g}] = \mathcal{O}[\mathbf{f}] \circ \mathcal{O}[\mathbf{g}]$. □

We would like to point out that $\mathbf{f}$ must be symmetric in order for $\mathcal{O}[\mathbf{f}]$ to be defined. It is not possible to extend $\mathcal{O}[-]$ to the dg-category of all thick maps for arbitrary $\mathcal{O}$. In what follows we will also extend the target of the Schur functor to get a dg-functor

$$\mathcal{O}_\bullet[-]\colon T_\Sigma(\mathscr{C}) \to T_\Sigma(\mathscr{C}).$$

Recall that the *tensor product* of two symmetric sequences $\mathcal{O}$ and $\mathcal{P}$ is the symmetric sequence $\mathcal{O} \otimes \mathcal{P}$ given by

$$(\mathcal{O} \otimes \mathcal{P})(n) = \bigoplus_{p+q=n} \operatorname{Ind}_{\Sigma_p\times\Sigma_q}^{\Sigma_n} \mathcal{O}(p) \otimes \mathcal{P}(q).$$

Here $\operatorname{Ind}_{\Sigma_p\times\Sigma_q}^{\Sigma_n} \mathcal{O}(p) \otimes \mathcal{P}(q)$ denotes the induced Σ_n-representation. This tensor product has the property that there is an isomorphism of functors from $\mathscr{C}$ to itself

$$(\mathcal{O} \otimes \mathcal{P})[-] \cong \mathcal{O}[-] \otimes \mathcal{P}[-],$$

and it makes the category of symmetric sequences into a symmetric monoidal dg-category, see [9, §2.1].

Definition 5.3. The *extended Schur dg-functor* $\mathcal{O}_\bullet[-]\colon T_\Sigma(\mathscr{C}) \to T_\Sigma(\mathscr{C})$ is defined as follows. For a symmetric thick map $\mathbf{f}\colon A \to B$, the n^{th} level $\mathcal{O}_n[\mathbf{f}]\colon \mathcal{O}[A]^{\otimes n} \to \mathcal{O}[B]^{\otimes n}$ of the thick map $\mathcal{O}_\bullet[\mathbf{f}]\colon \mathcal{O}[A] \to \mathcal{O}[B]$ is defined by requiring commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{O}[A]^{\otimes n} \arrow[r, "\mathcal{O}_n{[\mathbf{f}]}"] \arrow[d, "\cong"] & \mathcal{O}[B]^{\otimes n} \arrow[d, "\cong"] \\
\mathcal{O}^{\otimes n}[A] \arrow[r, "\mathcal{O}^{\otimes n}{[\mathbf{f}]}"] & \mathcal{O}^{\otimes n}[B]
\end{tikzcd}$$

The lower horizontal map is the application of the dg-functor $\mathcal{O}^{\otimes n}[-]\colon T_\Sigma(\mathscr{C}) \to \mathscr{C}$ from Proposition 5.2 to the symmetric thick map $\mathbf{f}$. The vertical maps are given by the natural isomorphism $\mathcal{O}[-]^{\otimes n} \cong \mathcal{O}^{\otimes n}[-]$ of functors from $\mathscr{C}$ to itself.

To be more explicit, observe that there is an isomorphism

$$\mathcal{O}[A]^{\otimes n} \cong \bigoplus_{r_1,\ldots,r_n\geq 0} (\mathcal{O}(r_1)\otimes\ldots\otimes\mathcal{O}(r_n))\otimes_{\Sigma_{r_1}\times\ldots\times\Sigma_{r_n}} A^{\otimes(r_1+\ldots+r_n)}$$

The map $\mathcal{O}_n[\mathbf{f}]$ acts only on the factor $A^{\otimes(r_1+\ldots+r_n)}$, and on the summand indexed by $(r_1,\ldots,r_n)$ it acts on this factor as $f_{r_1+\ldots+r_n}$. The thick map $\mathcal{O}_\bullet[\mathbf{f}]$ is symmetric because $\mathcal{O}\mapsto\mathcal{O}[-]$ is a symmetric monoidal functor ([9, Proposition 2.1.5]). $\mathcal{O}_\bullet[-]$ is a dg-functor because it is so at each level.

Proposition 5.4. *The extended Schur dg-functor $\mathcal{O}_\bullet[-]\colon T_\Sigma(\mathscr{C})\to T_\Sigma(\mathscr{C})$ takes morphisms to morphisms and pseudo-derivations to pseudo-derivations.*

Proof. Suppose that $\mathbf{h}\colon A\to A$ is a pseudo-derivation, and let $\mathbf{H}=\mathcal{O}_\bullet[\mathbf{h}]\colon\mathcal{O}[A]\to\mathcal{O}[A]$. We need to show that $\mathbf{H}$ is a pseudo-derivation. Indeed, for any p,q the restriction of the map

$$(H_p\otimes 1-1\otimes H_q)H_{p+q}\colon \mathcal{O}[A]^{\otimes(p+q)}\to\mathcal{O}[A]^{\otimes(p+q)}$$

to the summand indexed by $(r_1,\ldots,r_{p+q})$ acts on the right factor $A^{\otimes(r_1+\ldots+r_{p+q})}$ as

$$(h_i\otimes 1-1\otimes h_j)h_{i+j}$$

where $i=r_1+\ldots+r_p$ and $j=r_{p+1}+\ldots+r_{p+q}$. Since $\mathbf{h}$ is a pseudo-derivation, this is equal to $h_i\otimes h_j$. But this is exactly how the map $H_p\otimes H_q\colon\mathcal{O}[A]^{\otimes(p+q)}\to\mathcal{O}[A]^{\otimes(p+q)}$ restricted to the component indexed by $(r_1,\ldots,r_{p+q})$ acts on the right factor. Thus,

$$(H_p\otimes 1-1\otimes H_q)H_{p+q}=H_p\otimes H_q.$$

By the same argument

$$-H_{p+q}(H_p\otimes 1-1\otimes H_q)=H_p\otimes H_q.$$

The proof that the dg-functor $\mathcal{O}_\bullet[-]\colon T_\Sigma(\mathscr{C})\to T_\Sigma(\mathscr{C})$ takes morphisms to morphisms is similar. □

Theorem 5.5. *Suppose that $\mathbb{Q}\subseteq\Bbbk$. Let $\mathscr{D}$ be a contraction, and let $\mathcal{O}$ be any symmetric sequence. Then there is a symmetric thick contraction*

$$\mathcal{O}_\bullet[\mathscr{D}]\colon\ \begin{tikzcd} \mathcal{O}[A] \arrow[loop left, "\mathbf{H}"] \arrow[r, shift left, "\mathbf{F}"] & \mathcal{O}[B] \arrow[l, shift left, "\mathbf{G}"] \end{tikzcd}$$

where $\mathbf{F}$ and $\mathbf{G}$ are the morphisms induced by f and g, and where $\mathbf{H}=\mathcal{O}_\bullet[\mathbf{h}^\Sigma]$ is the pseudo-derivation induced by the symmetrized tensor trick homotopy (Definition 4.1).

Proof. Use Theorem 4.4 to extend the contraction $\mathscr{D}$ to a symmetric thick contraction $\underline{\mathscr{D}}^\Sigma$. Any dg-functor preserves contractions, so applying the extended Schur dg-functor $\mathcal{O}_\bullet[-]$ (Definition 5.3) to $\underline{\mathscr{D}}^\Sigma$ we obtain a new symmetric thick contraction

$$\mathcal{O}_\bullet[\mathscr{D}]\colon\ \begin{tikzcd} \mathcal{O}[A] \arrow[loop left, "\mathbf{H}"] \arrow[r, shift left, "\mathbf{F}"] & \mathcal{O}[B] \arrow[l, shift left, "\mathbf{G}"] \end{tikzcd}.$$

According to Theorem 4.4, $\mathbf{f}$ and $\mathbf{g}$ are morphisms and $\mathbf{h}^\Sigma$ is a pseudo-derivation, so by Proposition 5.4, $\mathbf{F}$ and $\mathbf{G}$ are morphisms and $\mathbf{H}$ is a pseudo-derivation. □

We can now show the necessity of the assumption $\mathbb{Q}\subseteq\Bbbk$.

Proposition 5.6. *If every contraction $\mathscr{D}$ can be extended to a symmetric thick contraction $\underline{\mathscr{D}}$ then necessarily $\mathbb{Q} \subseteq \mathbb{k}$.*

Proof. For integers n and m, let $D^n(m)$ denote the chain complex whose underlying graded $\mathbb{k}$-module has one generator x in degree n and one generator y in degree $n-1$, and where the differential is given by $d(x) = my$ and $d(y) = 0$. Defining $h\colon D^2(1) \to D^2(1)$ by $h(x) = 0$, $h(y) = x$, and $f = 0$, $g = 0$, we get a contraction

$$\mathscr{D}\colon\ \begin{tikzcd} D^2(1) \arrow[loop left, "h"] \arrow[r, shift left, "f"] & 0 \arrow[l, shift left, "g"] \end{tikzcd} .$$

If this had an extension to a symmetric thick contraction $\underline{\mathscr{D}}$, then we could apply $\mathcal{O}[-]$ to this, for any symmetric sequence $\mathcal{O}$. Consider the particular symmetric sequence $\mathcal{S}$ with $\mathcal{S}(0) = 0$ and $\mathcal{S}(n) = \mathbb{k}$, the trivial representation of Σ_n, for $n \geq 1$. The value at A of the associated Schur functor is the (non-unital) symmetric algebra on A:

$$\mathcal{S}[A] = \bigoplus_{n\geq 1} (A^{\otimes n})_{\Sigma_n}.$$

Applying $\mathcal{S}[-]$ to the symmetric thick contraction $\underline{\mathscr{D}}$, we would get a contraction

$$\begin{tikzcd} \mathcal{S}[D^2(1)] \arrow[loop left, "H"] \arrow[r, shift left, "F"] & \mathcal{S}[0] \arrow[l, shift left, "G"] \end{tikzcd} .$$

But $\mathcal{S}[0] = 0$, so this can only happen if $\mathcal{S}[D^2(1)]$ is contractible. As a graded module, $\mathcal{S}[D^2(1)]$ has basis x^{n+1}, $x^n y$, where $|x^{n+1}| = 2n+2$ and $|x^n y| = 2n+1$. The differential is given by $d(x^{n+1}) = (n+1)x^n y$ and $d(x^n y) = 0$, so there is a direct sum decomposition

$$\mathcal{S}[D^2(1)] \cong \bigoplus_{n\geq 0} D^{2n+2}(n+1)$$

Therefore, $\mathcal{S}[D^2(1)]$ is contractible if and only if $D^{2n+2}(n+1)$ is contractible for all $n \geq 0$. But $D^{2n+2}(n+1)$ is contractible if and only if $n+1$ is invertible in $\mathbb{k}$. Hence, $\mathcal{S}[D^2(1)]$ is contractible if and only if $\mathbb{Q} \subseteq \mathbb{k}$. $\square$

6. Perturbation lemma and tensor trick for algebras over operads

Recall that the *composition product* of two symmetric sequences $\mathcal{O}$ and $\mathcal{P}$ is the symmetric sequence

$$\mathcal{O} \circ \mathcal{P} = \bigoplus_{n\geq 0} \mathcal{O}(n) \otimes_{\Sigma_n} \mathcal{P}^{\otimes n}.$$

The composition product has the property that there is an isomorphism of functors from $\mathscr{C}$ to itself

$$(\mathcal{O} \circ \mathcal{P})[-] \cong \mathcal{O}[\mathcal{P}[-]],$$

and it makes the category of symmetric sequences into a monoidal category, see [9, §2.2]. An *operad* is a monoid in this monoidal category [9, §3.1]. If $\mathcal{O}$ is an operad then the associated Schur functor $\mathcal{O}[-]$ becomes a monad [29, Chapter VI], and an *algebra over* $\mathcal{O}$ is an algebra over the monad $\mathcal{O}[-]$, i.e., an object A together with a morphism $\gamma_A\colon \mathcal{O}[A] \to A$ satisfying a unit and an associativity constraint, see [9, §3.2], [29, p. 140].

Definition 6.1. Let $\mathcal{O}$ be an operad, and let A and B be $\mathcal{O}$-algebras. A *thick map of $\mathcal{O}$-algebras* is a symmetric thick map $\mathbf{f}\colon A \to B$ such that the diagram

$$\begin{tikzcd} \mathcal{O}[A] \arrow[r, "\mathcal{O}[\mathbf{f}]"] \arrow[d, "\gamma_A"] & \mathcal{O}[B] \arrow[d, "\gamma_B"] \\ A \arrow[r, "f_1"] & B \end{tikzcd}$$

commutes, where the upper horizontal map is defined as in Proposition 5.2.

If A is an $\mathcal{O}$-algebra, then every $\mu \in \mathcal{O}(n)$ gives rise to a map $\mu_A\colon A^{\otimes n} \to A$ of degree $|\mu|$. In more elementary terms, a thick map of $\mathcal{O}$-algebras is a sequence

$$\mathbf{f} = \{f_n\colon A^{\otimes n} \to B^{\otimes n}\}_{n\geq 0}$$

of Σ_n-equivariant maps of the same degree $|\mathbf{f}|$ such that

$$f_1\mu_A = (-1)^{|\mu||\mathbf{f}|}\mu_B f_n$$

for every $\mu \in \mathcal{O}(n)$.

Proposition 6.2. *If $\mathbf{f},\mathbf{g}\colon A \to B$ and $\mathbf{h}\colon B \to C$ are thick maps of $\mathcal{O}$-algebras, then so are $\mathbf{h}\circ\mathbf{f}$, $\partial(\mathbf{f})$ and $a\mathbf{f}+b\mathbf{g}$, for $a,b \in \Bbbk$. In other words, thick maps of $\mathcal{O}$-algebras form a dg-subcategory of $T_\Sigma(\mathscr{C})$.*

Proof. This is a direct consequence of the fact that the extended Schur functor $\mathcal{O}[-]\colon T_\Sigma(\mathscr{C}) \to \mathscr{C}$ is a dg-functor (Proposition 5.2) and that $\mathbf{f} \mapsto f_1$ is a dg-functor. □

Thick maps of $\mathcal{O}$-algebras are a simultaneous generalization of morphisms and derivations ([11, Definition 2.5],[28, §5.3.8]) of $\mathcal{O}$-algebras.

Proposition 6.3.
(1) *A morphism $f\colon A \to B$ in $\mathscr{C}$ is a morphism of $\mathcal{O}$-algebras if and only if the thick map $\mathbf{f}\colon A \to B$ given by $f_n = f^{\otimes n}$ is a thick map of $\mathcal{O}$-algebras.*
(2) *Let $l,r\colon A \to B$ be morphisms of $\mathcal{O}$-algebras. A map $d\colon A \to B$ is an (l,r)-derivation of $\mathcal{O}$-algebras if and only if the thick map $\mathbf{d}\colon A \to B$ given by*

$$d_n = \sum_{i+1+j=n} l^{\otimes i}\otimes d \otimes r^{\otimes j}$$

is a thick map of $\mathcal{O}$-algebras.

Proof. The proof is an exercise in the definitions. □

In our newly introduced language of thick maps of $\mathcal{O}$-algebras, Proposition 3.1 says that a contraction $\mathscr{D}$ is an algebra contraction if and only if it has an extension to a thick contraction $\underline{\mathscr{D}}$ where

(1) $\mathbf{f}$ and $\mathbf{g}$ are morphisms and $\mathbf{h}$ is a $(\mathbf{1},\mathbf{gf})$-derivation.
(2) $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are thick maps of $\mathcal{O}$-algebras, where $\mathcal{O}$ is the non-symmetric operad governing (not necessarily associative) binary algebras.

We cannot use (1) and (2) to define contractions of algebras over arbitrary operads $\mathcal{O}$, because asking a thick map to be simultaneously a $(\mathbf{1},\mathbf{gf})$-derivation and a thick map of $\mathcal{O}$-algebras would be asking too much. The first condition rules out symmetry whereas the latter demands it. However, as we saw in Theorem 3.4, the essential property of $(\mathbf{1},\mathbf{gf})$-derivations that makes the Algebra Perturbation

Lemma work is the pseudo-derivation property (Definition 3.2). Therefore, the following generalization of algebra contractions suggests itself.

Definition 6.4. Let $\mathcal{O}$ be an operad. A *thick contraction of $\mathcal{O}$-algebras* is a thick contraction

$$\underline{\mathscr{D}}\colon\ \mathbf{h} \circlearrowright A \underset{\mathbf{g}}{\overset{\mathbf{f}}{\rightleftarrows}} B$$

where $\mathbf{f}$ and $\mathbf{g}$ are morphisms and $\mathbf{h}$ is a pseudo-derivation, where A and B are $\mathcal{O}$-algebras and where $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are thick maps of $\mathcal{O}$-algebras.

Theorem 6.5 (Thick $\mathcal{O}$-algebra Perturbation Lemma). *Let $\underline{\mathscr{D}}$ be a thick contraction of $\mathcal{O}$-algebras and let $\mathbf{t}$ be a thick perturbation of A which is a derivation and a thick map of $\mathcal{O}$-algebras such that $\mathbf{1}-\mathbf{ht}$ is invertible. If we use the formulas of the Basic Perturbation Lemma (Theorem 1.3) to form $\mathbf{f}'$, $\mathbf{g}'$, $\mathbf{h}'$ and $\mathbf{t}'$, then $t = t_1$ and $t' = t'_1$ are perturbations of A and B and*

$$\underline{\mathscr{D}}^{\mathbf{t}}\colon\ \mathbf{h}' \circlearrowright A^{t} \underset{\mathbf{g}'}{\overset{\mathbf{f}'}{\rightleftarrows}} B^{t'}$$

is a thick contraction of $\mathcal{O}$-algebras.

Proof. By Theorem 3.4, $\underline{\mathscr{D}}^{\mathbf{t}}$ is a thick contraction, $\mathbf{f}'$ and $\mathbf{g}'$ are morphisms, $\mathbf{h}'$ is a pseudo-derivation, $\mathbf{t}'$ is a derivation, and $t = t_1$ and $t' = t'_1$ are perturbations of A and B. We need to verify that the perturbed objects A^t and $B^{t'}$ are $\mathcal{O}$-algebras and that $\mathbf{f}'$, $\mathbf{g}'$ and $\mathbf{h}'$ are thick maps of $\mathcal{O}$-algebras between A^t and $B^{t'}$.

Since $\mathbf{t}$ and $\mathbf{h}$ are thick maps of $\mathcal{O}$-algebras from A to itself, it follows from Proposition 6.2 that so are $\mathbf{1}-\mathbf{ht}$, the inverse $(\mathbf{1}-\mathbf{ht})^{-1}$, and $\mathbf{\Sigma} = \mathbf{t}(\mathbf{1}-\mathbf{ht})^{-1}$. Hence the perturbed maps $\mathbf{f}'$, $\mathbf{g}'$, $\mathbf{h}'$ and $\mathbf{t}'$, being built by composing and adding thick maps of $\mathcal{O}$-algebras, are again thick maps of $\mathcal{O}$-algebras, viewed as thick maps between A and B.

By Proposition 6.3 (2), since $\mathbf{t}\colon A \to A$ is a derivation and a thick map of $\mathcal{O}$-algebras, t is a derivation of $\mathcal{O}$-algebras, so A^t, which is just A with perturbed differential $d_A + t$, becomes an $\mathcal{O}$-algebra with the same structure maps as A. Similarly, since $\mathbf{t}'\colon B \to B$ is a derivation and a thick map of $\mathcal{O}$-algebras, $B^{t'}$ is an $\mathcal{O}$-algebra with the same structure maps as B. Since the $\mathcal{O}$-algebra structure maps for A^t and $B^{t'}$ are the same as those for A and B respectively, the thick maps $\mathbf{f}'$, $\mathbf{g}'$, $\mathbf{h}'$ and $\mathbf{t}'$ are indeed thick maps of $\mathcal{O}$-algebras between A^t and $B^{t'}$. □

Invertibility of $\mathbf{1}-\mathbf{ht}$ can be ensured by having suitable filtrations on the objects. Also, we will eventually want to use thick maps as a black box. This motivates the following definition.

Definition 6.6. A *filtered contraction of $\mathcal{O}$-algebras* is a filtered contraction

$$\mathscr{D}\colon\ h \circlearrowright A \underset{g}{\overset{f}{\rightleftarrows}} B$$

which has an extension to a thick contraction of $\mathcal{O}$-algebras $\underline{\mathscr{D}}$ in which the maps f_n, g_n and h_n preserve the induced filtrations on $A^{\otimes n}$ and $B^{\otimes n}$.

By definition, the p^{th} level of the induced filtration on $A^{\otimes n}$ is the image of the sum of all maps $F_{i_1}A \otimes \ldots \otimes F_{i_n}A \to A^{\otimes n}$ where $i_1 + \ldots + i_n = p$.

If we spell out Definition 6.6 without using the language of thick maps, we arrive at the equivalent definition stated in the introduction.

Definition 6.7 (Alternative definition of $\mathcal{O}$-algebra contractions)**.** A contraction of $\mathcal{O}$-algebras is a contraction $\mathscr{D}$ in which A and B are $\mathcal{O}$-algebras, f and g are morphisms of $\mathcal{O}$-algebras, and there exists a sequence of homotopies $\{h_n\colon A^{\otimes n} \to A^{\otimes n}\}_{n\geq 1}$ with $h_1 = h$ such that

(1) For each $n \geq 1$, the diagram

$$\mathscr{D}_n\colon \; h_n \circlearrowright A^{\otimes n} \underset{g^{\otimes n}}{\overset{f^{\otimes n}}{\rightleftarrows}} B^{\otimes n}$$

is a contraction.

(2) For all $\mu \in \mathcal{O}(n)$, we have $\mu_A h_n = (-1)^{|\mu|} h \mu_A$.

(3) For all $p, q \geq 1$, there is an equality of maps from $A^{\otimes p+q}$ to itself

$$(h_p \otimes 1 - 1 \otimes h_q) h_{p+q} = -h_{p+q}(h_p \otimes 1 - 1 \otimes h_q) = h_p \otimes h_q.$$

We say that $\mathscr{D}$ is a *filtered contraction of $\mathcal{O}$-algebras* if, in addition, A and B are equipped with bounded below exhaustive filtrations such that $\mathscr{D}_n$ is a filtered contraction for all n, where $A^{\otimes n}$ and $B^{\otimes n}$ are given the induced filtrations.

We can now give the proof of Theorem A.

Theorem 6.8 ($\mathcal{O}$-algebra Perturbation Lemma)**.** *Let $\mathscr{D}$ be a filtered contraction of $\mathcal{O}$-algebras. If t is a perturbation of A which is a derivation of $\mathcal{O}$-algebras and which lowers the filtration on A, then $\mathscr{D}^t$ is a filtered contraction of $\mathcal{O}$-algebras.*

Proof. By definition of a filtered contraction of $\mathcal{O}$-algebras (Definition 6.6), there exists a thick contraction of $\mathcal{O}$-algebras $\underline{\mathscr{D}}$ which extends $\mathscr{D}$ and where the maps f_n, g_n, h_n preserve the induced filtrations on $A^{\otimes n}$ and $B^{\otimes n}$. Since t is a derivation of $\mathcal{O}$-algebras the symmetric thick map $\mathbf{t}$ defined by $t_n = \sum_{i+1+j=n} 1^{\otimes i} \otimes t \otimes 1^{\otimes j}$ is a thick map of $\mathcal{O}$-algebras by Proposition 6.3. Moreover, since t lowers the filtration on A, each t_n lowers the induced filtration on $A^{\otimes n}$. Therefore, the series $\sum_{m\geq 0}(\mathbf{ht})^m$ is a well defined thick map of $\mathcal{O}$-algebras and a filtration preserving inverse of $\mathbf{1} - \mathbf{ht}$. Now we can apply Theorem 6.5 to get a thick contraction of $\mathcal{O}$-algebras $\underline{\mathscr{D}}^{\mathbf{t}}$. The perturbed maps are filtration preserving since $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are filtration preserving and since $\mathbf{\Sigma} = \sum_{m\geq 0} \mathbf{t}(\mathbf{ht})^m$ lowers the filtration. Thus, the thick contraction of $\mathcal{O}$-algebras $\underline{\mathscr{D}}^{\mathbf{t}}$ exhibits its first level $\mathscr{D}^t$ as a filtered contraction of $\mathcal{O}$-algebras. □

Having proved Theorem A, let us revisit the 'Tensor Trick' for Schur functors (Theorem 5.5) and examine how the algebraic structure on $\mathcal{O}[A]$ comes into play when $\mathcal{O}$ is an operad.

Proposition 6.9. *If $\mathcal{O}$ is an operad, then for any symmetric thick map $\mathbf{f}\colon A \to B$, the thick map $\mathcal{O}_\bullet[\mathbf{f}]\colon \mathcal{O}[A] \to \mathcal{O}[B]$ is a thick map of $\mathcal{O}$-algebras.*

Proof. Let $\mathbf{F} = \mathcal{O}_\bullet[\mathbf{f}]$. It is straightforward to check that the diagram

$$\begin{tikzcd}
\mathcal{O}[\mathcal{O}[A]] \arrow[r, "\mathcal{O}[\mathbf{F}]"] \arrow[d, "\cong"] & \mathcal{O}[\mathcal{O}[B]] \arrow[d, "\cong"] \\
(\mathcal{O} \circ \mathcal{O})[A] \arrow[r, "(\mathcal{O}\circ\mathcal{O})[\mathbf{f}]"] & (\mathcal{O} \circ \mathcal{O})[B]
\end{tikzcd}$$

commutes. Since the $\mathcal{O}$-algebra structure on $\mathcal{O}[A]$ is given by the composite

$$\mathcal{O}[\mathcal{O}[A]] \xrightarrow{\cong} (\mathcal{O} \circ \mathcal{O})[A] \xrightarrow{\gamma_{\mathcal{O}}[A]} \mathcal{O}[A] \ ,$$

see [9, §3.2.13], this implies that $\mathbf{F}$ is a thick map of $\mathcal{O}$-algebras. □

We are now in position to give the proof of Theorem B, but let us first specify what filtrations we are using. For any chain complex A, by the *weight filtration* on $\mathcal{O}[A]$ we mean the bounded below filtration $0 = F_{-1}A \subseteq F_0 A \subseteq \ldots \subseteq \mathcal{O}[A]$ where

$$F_p A = \bigoplus_{n \leq p} \mathcal{O}(n) \otimes_{\Sigma_n} A^{\otimes n}.$$

Theorem 6.10 ($\mathcal{O}$-algebra Tensor Trick)**.** *Suppose that $\mathbb{Q} \subseteq \Bbbk$. Let $\mathscr{D}$ be a contraction. Then, with respect to the weight filtrations on $\mathcal{O}[A]$ and $\mathcal{O}[B]$, there is a filtered contraction of $\mathcal{O}$-algebras*

$$\mathcal{O}[\mathscr{D}]\colon \begin{tikzcd} \mathcal{O}[A] \arrow[loop left, "H"] \arrow[r, shift left, "F"] & \mathcal{O}[B] \arrow[l, shift left, "G"] \end{tikzcd} .$$

Here F and G are the morphisms of $\mathcal{O}$-algebras induced by f and g, and the homotopy H is induced by the symmetrized tensor trick homotopy.

Proof. By Theorem 5.5 the contraction $\mathscr{D}$ gives rise to a symmetric thick contraction

$$\mathcal{O}_\bullet[\mathscr{D}]\colon \begin{tikzcd} \mathcal{O}[A] \arrow[loop left, "\mathbf{H}"] \arrow[r, shift left, "\mathbf{F}"] & \mathcal{O}[B] \arrow[l, shift left, "\mathbf{G}"] \end{tikzcd}$$

where $\mathbf{F} = \mathcal{O}_\bullet[\mathbf{f}]$, $\mathbf{G} = \mathcal{O}_\bullet[\mathbf{g}]$, $\mathbf{H} = \mathcal{O}_\bullet[\mathbf{h}^\Sigma]$, and where $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}^\Sigma$ are as in Theorem 4.4. Moreover, by the same theorem $\mathbf{F}$ and $\mathbf{G}$ are morphisms and $\mathbf{H}$ is a pseudo-derivation. By examining the definitions of these maps, one sees that they preserve the weight filtrations. By Proposition 6.9, $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ are thick maps of $\mathcal{O}$-algebras. Hence, $\mathcal{O}_\bullet[\mathscr{D}]$ is a filtered thick contraction of $\mathcal{O}$-algebras, the first level of which is the desired filtered contraction of $\mathcal{O}$-algebras $\mathcal{O}[\mathscr{D}]$. □

7. Perturbation lemma and tensor trick for coalgebras over cooperads

In this section we will dualize the results of the previous section. The proofs are virtually the same and will therefore be omitted.

Recall that a *cooperad* $\mathcal{C}$ is a comonoid in the monoidal category of symmetric sequences with the composition product, see [10, §1.2.17],[11, §1.7] or [28, §4.7]. If $\mathcal{C}$ is a cooperad, then the associated Schur functor $\mathcal{C}[-]$ becomes a comonad [29, p. 139], and a *$\mathcal{C}$-coalgebra* is a coalgebra over this comonad, i.e., an object A together with a morphism $\Delta_A\colon A \to \mathcal{C}[A]$ satisfying a coassociativity constraint, see [10, §1.2.17],[11, §1.7] or [28, §4.7.4].

Definition 7.1. Let $\mathcal{C}$ be a cooperad and let A and B be $\mathcal{C}$-coalgebras. A *thick map of $\mathcal{C}$-coalgebras* is a symmetric thick map $\mathbf{f}\colon A \to B$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "\Delta_A"] & B \arrow[d, "\Delta_B"] \\ \mathcal{C}[A] \arrow[r, "\mathcal{C}[\mathbf{f}]"] & \mathcal{C}[B] \end{tikzcd}$$

commutes.

Proposition 7.2. *If $\mathbf{f}, \mathbf{g}\colon A \to B$ and $\mathbf{h}\colon B \to C$ are thick maps of $\mathcal{C}$-coalgebras, then so are $\mathbf{h} \circ \mathbf{f}$, $\partial(\mathbf{f})$ and $a\mathbf{f} + b\mathbf{g}$, for $a, b \in \Bbbk$.*

Proposition 7.3.
1. *A morphism $f\colon A \to B$ is a morphism of $\mathcal{C}$-coalgebras if and only if the thick map $\mathbf{f}\colon A \to B$ given by $f_n = f^{\otimes n}$ is a thick map of $\mathcal{C}$-coalgebras.*
2. *Let $l, r\colon A \to B$ be morphisms of $\mathcal{C}$-coalgebras. A map $d\colon A \to B$ is an (l, r)-coderivation of $\mathcal{C}$-coalgebras if and only if the thick map $\mathbf{d}\colon A \to B$ given by*
$$d_n = \sum_{i+1+j=n} l^{\otimes i} \otimes d \otimes r^{\otimes j}$$
is a thick map of $\mathcal{C}$-coalgebras.

Definition 7.4. A *thick contraction of $\mathcal{C}$-coalgebras* is a thick contraction

$$\underline{\mathscr{D}}\colon \begin{tikzcd} A \arrow[loop left, "\mathbf{h}"] \arrow[r, shift left, "\mathbf{f}"] & B \arrow[l, shift left, "\mathbf{g}"] \end{tikzcd}$$

where $\mathbf{f}$ and $\mathbf{g}$ are morphisms and $\mathbf{h}$ is a pseudo-derivation, where A and B are $\mathcal{C}$-coalgebras and where $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are thick maps of $\mathcal{C}$-coalgebras.

Theorem 7.5 (Thick $\mathcal{C}$-coalgebra Perturbation Lemma)**.** *If $\underline{\mathscr{D}}$ is a thick contraction of $\mathcal{C}$-coalgebras and if the perturbation $\mathbf{t}$ is a derivation and a thick map of $\mathcal{C}$-coalgebras such that $\mathbf{1} - \mathbf{ht}$ is invertible then $\underline{\mathscr{D}}^{\mathbf{t}}$ is a thick contraction of $\mathcal{C}$-coalgebras.*

Definition 7.6. A *filtered contraction of $\mathcal{C}$-coalgebras* is a filtered contraction $\mathscr{D}$ which has an extension to a thick contraction of $\mathcal{C}$-coalgebras $\underline{\mathscr{D}}$ in which the maps f_n, g_n and h_n preserve the induced filtrations on $A^{\otimes n}$ and $B^{\otimes n}$.

Theorem 7.7 ($\mathcal{C}$-coalgebra Perturbation Lemma)**.** *If $\mathscr{D}$ is a filtered contraction of $\mathcal{C}$-coalgebras and if the perturbation t is a coderivation of $\mathcal{C}$-coalgebras which lowers the filtration on A then $\mathscr{D}^t$ is a filtered contraction of $\mathcal{C}$-coalgebras.*

Theorem 7.8 ($\mathcal{C}$-coalgebra Tensor Trick)**.** *Suppose that $\mathbb{Q} \subseteq \Bbbk$. Let $\mathscr{D}$ be a contraction. Then, with respect to the weight filtrations on $\mathcal{C}[A]$ and $\mathcal{C}[B]$, there is a filtered contraction of $\mathcal{C}$-coalgebras*

$$\mathcal{C}[\mathscr{D}]\colon \begin{tikzcd} \mathcal{C}[A] \arrow[loop left, "H"] \arrow[r, shift left, "F"] & \mathcal{C}[B] \arrow[l, shift left, "G"] \end{tikzcd}$$

where F and G are the morphisms of $\mathcal{C}$-coalgebras induced by f and g, and where H is induced by the symmetrized tensor trick homotopy.

8. Application: Transferring $\mathcal{O}_\infty$-algebra structures

In this section we will apply our main results to transfer $\mathcal{O}_\infty$-algebra structures, where $\mathcal{O}$ is a Koszul operad. For the Koszul duality theory of operads, we refer the reader to [12],[11],[10],[28].

Let $\mathcal{O}$ be a Koszul operad. The operad $\mathcal{O}_\infty$ which governs 'strongly homotopy' $\mathcal{O}$-algebras is by definition the cobar construction ([11, §2.1],[10, §3.1.10],[28, 5.5.4]) $\Omega(\mathcal{O}^{\text{i}})$ on the Koszul dual cooperad $\mathcal{O}^{\text{i}}$ ([10, 5.2.8],[28, §6.2.1]), see [11, Definition 4.14],[28, §8.1.1]. The associative operad $\mathcal{A}s$, the commutative operad $\mathcal{C}om$ and the

Lie operad $\mathcal{L}ie$ are examples of Koszul operads. The associated strongly homotopy algebras are respectively A_∞-algebras [35], C_∞-algebras [11, §5.3] and L_∞-algebras [26]. The key to proving transfer theorems using homological perturbation theory is the following proposition.

Proposition 8.1 ([11, Proposition 2.15],[28, Proposition 8.1.18])**.** *Let $\mathcal{O}$ be a Koszul operad and let A be a chain complex. There is a one-to-one correspondence between $\mathcal{O}_\infty$-algebra structures on A and $\mathcal{O}^{\text{i}}$-coderivation perturbations t of the $\mathcal{O}^{\text{i}}$-coalgebra $\mathcal{O}^{\text{i}}[A]$ which lower the weight filtration.*

Specializing this proposition to $\mathcal{O} = \mathcal{A}s$ (where $\mathcal{A}s^! = \mathcal{A}s$), one recovers the well known fact that A_∞-algebra structures on a chain complex A correspond to weight decreasing coderivation perturbations of the tensor coalgebra $T(sA)$, see [35]. Specializing to $\mathcal{O} = \mathcal{L}ie$ (where $\mathcal{L}ie^! = \mathcal{C}om$), one recovers the correspondence between L_∞-algebra structures on a chain complex A and weight decreasing coderivation perturbations of the symmetric coalgebra $\Lambda(sA)$, see [26].

Theorem 8.2 (Transfer theorem for $\mathcal{O}_\infty$-algebras)**.** *Suppose that $\mathbb{Q} \subseteq \Bbbk$. Let $\mathcal{O}$ be a Koszul operad and let*

$$\mathscr{D}\colon\ h \circlearrowright A \underset{g}{\overset{f}{\rightleftarrows}} B$$

be a contraction. Given an $\mathcal{O}_\infty$-algebra structure ρ on A, there is an $\mathcal{O}_\infty$-algebra structure ρ' on B and quasi-isomorphisms of $\mathcal{O}_\infty$-algebras

$$(A,\rho) \overset{\sim}{\rightleftarrows} (B,\rho')$$

Proof. Apply the $\mathcal{O}^{\text{i}}$-coalgebra tensor trick (Theorem 7.8) to the contraction $\mathscr{D}$ to obtain a filtered contraction of $\mathcal{O}^{\text{i}}$-coalgebras

$$H \circlearrowright \mathcal{O}^{\text{i}}[A] \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{O}^{\text{i}}[B]\ .$$

Next, use Proposition 8.1 to reinterpret the $\mathcal{O}_\infty$-algebra structure ρ on A as a weight decreasing $\mathcal{O}^{\text{i}}$-coderivation perturbation t on $\mathcal{O}^{\text{i}}[A]$. Then we can use the $\mathcal{O}^{\text{i}}$-coalgebra Perturbation Lemma (Theorem 7.7) to obtain a new filtered contraction of $\mathcal{O}^{\text{i}}$-coalgebras

$$H' \circlearrowright \mathcal{O}^{\text{i}}[A]^t \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{O}^{\text{i}}[B]^{t'}$$

where the induced perturbation t' is an $\mathcal{O}^{\text{i}}$-coderivation which lowers the weight filtration on $\mathcal{O}^{\text{i}}[B]$, and the maps F' and G' are morphisms of $\mathcal{O}^{\text{i}}$-coalgebras. Using Proposition 8.1 again, we may reinterpret t' as an $\mathcal{O}_\infty$-algebra structure ρ' on B. As morphisms of chain complexes, F' and G' are inverse homotopy equivalences so in particular they are quasi-isomorphisms. Since F' and G' are also morphisms of $\mathcal{O}^{\text{i}}$-coalgebras, they correspond to quasi-isomorphisms between the two $\mathcal{O}_\infty$-algebras (A,ρ) and (B,ρ'). □

Corollary 8.3 (Minimality theorem for $\mathcal{O}_\infty$-algebras)**.** *Suppose that $\Bbbk$ is a field of characteristic zero. Let $\mathcal{O}$ be a Koszul operad and let A be a chain complex. Suppose given an $\mathcal{O}_\infty$-algebra structure ρ on A. Then there is an $\mathcal{O}_\infty$-algebra structure ρ'*

on the homology $\mathrm{H}_*(A)$*, with trivial differential, and quasi-isomorphisms of* $\mathcal{O}_\infty$*-algebras*

$$\begin{tikzcd} A \arrow[r, shift left, "\sim"] & \mathrm{H}_*(A) \arrow[l, shift left] \end{tikzcd}$$

Proof. Since we have field coefficients, it is possible to find a contraction of chain complexes

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, shift left, "f"] & \mathrm{H}_*(A) \arrow[l, shift left, "g"] \end{tikzcd}$$

where the homology $\mathrm{H}_*(A)$ is given the zero differential. Then apply the previous theorem with $B = \mathrm{H}_*(A)$. □

Remark 8.4. If $\mathcal{O}$ is a non-symmetric Koszul operad then one can drop the assumption that $\mathbb{Q} \subseteq \Bbbk$ in Theorem 8.2 and Corollary 8.3.

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